# Tightly Circumscribed Regular Polygons

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A regular polygon circumscribing another regular polygon (with a different side number) may be tightened to minimize the difference of both areas. The manuscripts computes the optimum result under the restriction that both polygons are concentric, and obtains limits if the process is repeated in a two-dimensional Babuschka-doll fashion with side numbers increasing or decreasing by one or stepping through the prime numbers. The new aspect compared to the circumscription discussed in the literature so far is that further squeezing of the outer polygon is possible as we drop the requirement of drawing intermediate spacing circles between the polygon pairs.

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## I. INTRODUCTION

### A. Notation

A regular $n$-gon is drawn with $n$ edges of some common side length $s_n$. The perimeter is $ns_n$. The incircle with inradius $r_n^{(i)}$ touches each edge at the mid point. Each edge covers an angle of

$$\phi_n = \frac{2\pi}{n} \tag{1}$$

if viewed from the incircle center. The area of the polygon is comprised of $n$ rotated copies of an isosceles triangle in which the short edge has length $s_n$, facing the angle $\phi_n$, and the two other edges have length $r_n^{(o)}$, which also is the radius of the circumcircle. This isosceles triangle might be sliced into two symmetric rectilinear triangles by drawing a line (apothem) from its base center to the midpoint of the polygon's incircle; the definition of the tangent and sine functions in these yield

$$\tan\frac{\phi_n}{2} = \frac{s_n/2}{r_n^{(i)}}, \tag{2}$$

$$\sin\frac{\phi_n}{2} = \frac{s_n/2}{r_n^{(o)}}, \tag{3}$$

and therefore

$$r_n^{(i)} = \frac{s_n}{2\tan\frac{\phi_n}{2}}, \tag{4}$$

$$r_n^{(o)} = \frac{s_n}{2\sin\frac{\phi_n}{2}}. \tag{5}$$

As a bridge between two-dimensional geometry and numerical algebra, we define the *standard* position of the polygon in the Cartesian $(x, y)$ plane by mapping $x$ and $y$ to the real and imaginary part of the complex plane, placing the vertices labeled $j = 0, 1, \ldots n-1$ counter-clock-wise at the coordinates

$$x + iy = r_n^{(o)} e^{2\pi j i/n}, \tag{6}$$

where $i \equiv \sqrt{-1}$ is the imaginary unit. Edges/sides are also enumerated from 0 to $n-1$ by calling the smaller of the two vertex labels that are joined.

### B. Tight Circumscription

The circumscription of a regular $n$-gon by a regular $m$-gon has been constructed earlier by an elementary step drawing the $n$-gon, its circumcircle, declaring this circle to be also the incircle of the $m$-gon with $r_m^{(i)} = r_n^{(o)}$, and drawing the $m$-gon around its incircle [1, p. 428][2, p. 2300][3, A051762].

The theme of this manuscript is to drop the requirement of equating the two circles and to search for smaller circumscribing regular $m$-gons in the extended range $r_n^{(i)} \le r_m^{(i)} \le r_n^{(o)}$.

This requires positions where some edges of the $m$-gon cut through the circumcircle of the $n$-gon. The tighter solution, however, may exist only within a restricted range of (relative) orientations of the polygons. The manuscript works out a full representation of the smallest circumscribing polygons, using the ratio $r_m^{(o)}/r_n^{(o)}$ as a figure of merit.

In overview, the achievable size ratios are calculated in Section II for concentric polygon pairs aligned such that the common center, a vertex of the inner polygon and a vertex of the outer polygon are collinear (standard positions). In Section III further size reductions of the outer polygon are found if the side number $n$ of the inner polygon is even and the outer polygon is turned around the common center. A cursory outlook in Section IV shows that shifting the center of the outer polygon away from the center of the inner polygon may define even smaller circumscribing $m$-gons.

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## II. CONCENTRIC STANDARD PLACEMENTS

### A. General side numbers

The definition of circumscription implies that the outer $m$-gon must stay further away from the origin than the inner $n$-gon at all viewing directions; at one or more points of contact, both polygons have the same distance $r_n^{(o)}$ to the origin.

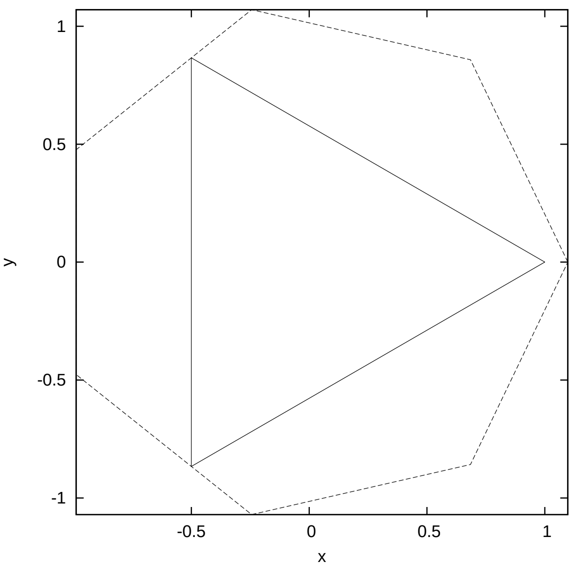


FIG. 1. Circumscribing a 3-gon by a 7-gon.

Finding the smallest $m$-gon for small side numbers like Fig. 1 works as follows. Any point on the side of the outer polygon of unit radius has a position $\exp(2\pi i j_o/m) + t[\exp(2\pi i(j_o{+}1)/m) - \exp(2\pi i j_o/m)]$ with parameter $0 \le t \le 1$. Once the vertex number $j_i$ of the inner polygon and the side number $j_o$ of the outer polygon which it touches are known, the point of contact between both polygons in the complex plane solves

$$r_m^{(o)}\{e^{2\pi i j_o/m} + t[e^{2\pi i(j_o+1)/m} - e^{2\pi i j_o/m}]\} = r_n^{(o)} e^{2\pi i j_i/n}. \tag{7}$$

and after division through $r_m^{(0)} e^{2\pi i j_0/m}$

$$1 + t[e^{2\pi i/m} - 1] = \frac{r_n^{(o)}}{r_m^{(o)}} e^{2\pi i(j_i/n - j_o/m)}. \tag{8}$$

Real and imaginary part of this equation establish an inhomogeneous $2\times 2$ linear system of equations for the unknown $t$ and $r_n^{(o)}/r_m^{(o)}$:

$$1 + t[\cos(2\pi/m) - 1] = \frac{r_n^{(o)}}{r_m^{(o)}} \cos[2\pi(j_i/n - j_o/m)], \tag{9}$$

$$t\sin(2\pi/m) = \frac{r_n^{(o)}}{r_m^{(o)}} \sin[2\pi(j_i/n - j_o/m)]. \tag{10}$$

$$\begin{pmatrix} \cos[2\pi(j_i/n - j_o/m)] & 1-\cos(2\pi/m) \\ \sin[2\pi(j_i/n - j_o/m)] & -\sin(2\pi/m) \end{pmatrix} \cdot \begin{pmatrix} r_n^{(o)}/r_m^{(o)} \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

The solution is obtained with Cramer's rule. The inverse ratio is [4, 4.3.16,4,3,35]

$$\begin{aligned} r_m^{(o)}/r_n^{(o)} &= \frac{\sin(j_i\phi_n - j_o\phi_m) - \sin[j_i\phi_n - (j_o+1)\phi_m]}{\sin\phi_m} \\ &= \frac{\cos[j_i\phi_n - (j_o+1/2)\phi_m]}{\cos(\phi_m/2)} \\ &= \frac{\cos[\frac{\pi}{nm}\{2j_i m - (2j_o+1)n\}]}{\cos(\pi/m)}. \end{aligned} \tag{12}$$

Let $d = m - n$ be the difference in the side numbers; then

$$2j_i m - (2j_o+1)n = 2j_i d - (2j_o - 2j_i + 1)n \tag{13}$$

in the argument of the cosine is a "mismatch" value in the angular directions.

The geometric interpretation of this equation: The minimum radius of the outer polygon is determined by the edge $j_o$ that first hits a nearby vertex $j_i$ while shrinking. The relevant index pair is the one that maximizes the cosine in the numerator, so the phase angle is steered towards zero or $2\pi$, equivalent to $j_i\phi_n \approx (j_0 + 1/2)\phi_m$. This means the relevant phase angle in the complex plane and viewing direction is where the vertex $j_i$ points near the middle of edge $j_o$.

Examples:

- In Fig. 1 we have set $n = 3$, $m = 7$, $r_n^{(o)} = 1$ and observe that $j_o = 2$, $j_i = 1$. Eq. (12) obtains $r_m^{(o)}/r_n^{(o)} = \cos(\pi/21)/\cos(\pi/7) \approx 1.097519$.
- In the inner pair of Fig. 2 we have set $n = 3$, $m = 4$, $r_n^{(o)} = 1$ and observe that $j_o = 1$, $j_i = 1$. Eq. (12) obtains $r_m^{(o)}/r_n^{(o)} = \sqrt{2}\cos(\pi/12) = (1+\sqrt{3})/2 \approx 1.366025$ [5, 6].
- In the inner pair of Fig. 4 we have set $n = 3$, $m = 5$, $r_n^{(o)} = 1$ and observe that $j_o = 1$, $j_i = 1$. The equation yields $r_m^{(o)}/r_n^{(o)} = \cos(\pi/15)/\cos(\pi/5) = \frac{3-\sqrt{5}}{4} + \sqrt{3}\sqrt{\frac{5-\sqrt{5}}{8}} \approx 1.2090569$.

Numerical examples of the size ratios of are gathered in Table I. [In solutions with interlaced circles—summarized in Section I B—the ratio $r_m^{(o)}/r_n^{(o)}$ always equals $1/\cos(\phi_m/2)$ derived with Eqs. (4) and (5). This restricted search space would have put constant values down each column.]

Where $m$ is a multiple of $n$, the table entries equal one. In these cases one can re-use $n$ vertices of the inner polygon as vertices of the outer polygon, and obtains its remaining $m - n$ vertices by regular subdivision of the angle, $\phi(m) = \frac{\phi(n)}{m/n}$. The circumradii are the same, $r_m^{(o)} = r_n^{(o)}$, and their ratio equals one.

| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 3 | 1.00000000 | 1.36602540 | 1.20905693 | 1.00000000 | 1.09751942 | 1.07313218 | 1.00000000 | 1.04570220 |
| 4 | 2.00000000 | 1.00000000 | 1.23606798 | 1.15470054 | 1.10991626 | 1.00000000 | 1.06417777 | 1.05146222 |
| 5 | 1.95629520 | 1.39680225 | 1.00000000 | 1.14837497 | 1.10544807 | 1.07905555 | 1.06158549 | 1.00000000 |
| 6 | 2.00000000 | 1.36602540 | 1.23606798 | 1.00000000 | 1.10991626 | 1.07313218 | 1.06417777 | 1.04570220 |
| 7 | 1.97766165 | 1.40532128 | 1.23109193 | 1.15147176 | 1.00000000 | 1.08068940 | 1.06285492 | 1.05040347 |
| 8 | 2.00000000 | 1.41421356 | 1.23606798 | 1.15470054 | 1.10991626 | 1.00000000 | 1.06417777 | 1.05146222 |
| 9 | 1.87938524 | 1.40883205 | 1.23305698 | 1.13715804 | 1.10853655 | 1.08136200 | 1.00000000 | 1.05082170 |
| 10 | 2.00000000 | 1.39680225 | 1.23606798 | 1.14837497 | 1.10991626 | 1.07905555 | 1.06417777 | 1.00000000 |

TABLE I. Size ratios $r_m^{(o)}/r_n^{(o)}$ as a function of the polygon edge count (rows $n \ge 3$ and columns $m \ge 3$) for the tight regular concentric positions.

### B. Consecutive side numbers

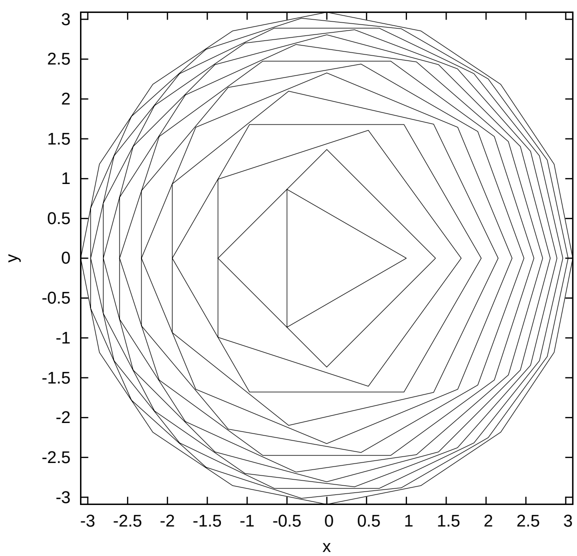


FIG. 2. Circumscribing a 3-gon by a 4-gon by a 5-gon ... up to a 16-gon, all at concentric standard positions.

The radial growth of a repeated, possibly infinite nesting is illustrated in Figure 2. The circumradius is the partial products of the first upper sub-diagonal of Table I.

The heuristics is here with $m = n + 1$ that

- for odd $n$, the edge $j_o = m/2 - 1$ hits the vertex $j_i = m/2 - 1$ with residual mismatch (13) equal to $-1$. [This is the best possible absolute value because $2j_i m$ is even and $(2j_o + 1)n$ is odd then.] So (12) is

$$r_{n+1}^{(o)}/r_n^{(o)} = \frac{\cos\frac{\pi}{n(n+1)}}{\cos\frac{\pi}{n+1}}, \quad n\,\text{odd}, \tag{14}$$

- For even $n$ and $m = n + 1$, the edge $j_o = n/2$ hits the vertex $j_i = n/2$ on the negative real axis with residual mismatch (13) equal to zero. So (12) yields

$$r_{n+1}^{(o)}/r_n^{(o)} = \frac{1}{\cos\frac{\pi}{n+1}}, n\,\text{even}. \tag{15}$$

These two equations constitute the first upper diagonal of Table I. Fencing the polygons up to infinity defines the limiting radius as an alternating product of these two factors,

$$\begin{aligned}\frac{r_\infty^{(o)}}{r_3^{(o)}} &= \frac{r_4^{(o)}}{r_3^{(o)}} \times \frac{r_5^{(o)}}{r_4^{(o)}} \times \frac{r_6^{(o)}}{r_5^{(o)}} \times \cdots \\ &= \prod_{n=3,5,7,\ldots} \frac{\cos\frac{\pi}{n(n+1)}}{\cos\frac{\pi}{n+1}} \prod_{n=4,6,8,\ldots} \frac{1}{\cos\frac{\pi}{n+1}} \\ &= \frac{\prod_{n=3,5,7,\ldots} \cos\frac{\pi}{n(n+1)}}{\prod_{n=3}^{\infty} \cos\frac{\pi}{n+1}} \\ &= \frac{1}{2K'} \prod_{n=3,5,7,\ldots} \cos\frac{\pi}{n(n+1)} \\ &\approx 4.16674437148793 \end{aligned} \tag{16}$$

where we have inserted (A21) and [1][3, A085365]

$$K' \equiv \prod_{n=3}^{\infty} \cos\frac{\pi}{n} \approx 0.1149420448532962007. \tag{17}$$

In Figure 3 the edge count of the circumscribed polygon is *decreased* from 16 to 3. The ratio of the circumradii of the 3-gon and the 16-gon is $\approx 6.2$ in the image. The observation is that here for $m = n - 1$ and

- $n$ even, the outer edge $j_o = n/2 - 1$ and $j_i = n/2$. The value of (12) is

$$r_{n-1}^{(o)}/r_n^{(o)} = \frac{1}{\cos\frac{\pi}{n-1}}, \quad n\,\text{even}. \tag{18}$$

- whereas for $n$ odd $j_o = (n-3)/2$ and $j_i = (n-1)/2$. The value of (12) is

$$r_{n-1}^{(o)}/r_n^{(o)} = \frac{\cos\frac{\pi}{n(n-1)}}{\cos\frac{\pi}{n-1}}, \quad n\,\text{odd} \tag{19}$$

The alternating infinite product of these terms, the finite radius of the free inner region in Figure 3 if inscribing indefinitely:

$$
\begin{aligned}
r_3^{(o)}/r_\infty^{(o)} &= \frac{r_3^{(o)}}{r_4^{(o)}} \times \frac{r_4^{(o)}}{r_5^{(o)}} \times \frac{r_5^{(o)}}{r_6^{(o)}} \times \cdots \\
&= \prod_{n=4,6,8,\ldots} \frac{r_{n-1}^{(o)}}{r_n^{(o)}} \times \frac{r_n^{(o)}}{r_{n+1}^{(o)}} \\
&= \prod_{n=4,6,8\ldots} \frac{1}{\cos\frac{\pi}{n-1}} \times \frac{\cos\frac{\pi}{(n+1)n}}{\cos\frac{\pi}{n}} \\
&\approx 8.5526818319553. \qquad (20)
\end{aligned}
$$

This is slightly smaller than the equivalent polygon circumscribing constant $1/K' \approx 8.7000366\ldots$ [3, A051762][2, p. 2300][1, 7, p. 428] by the factor

$$
\prod_{n=4,6,8,\ldots} \cos\frac{\pi}{n(n+1)} \approx 0.98306273874458351\ldots, \qquad (21)
$$

based on (17). The areas have been smaller relative to the published construction with interspersed circles. The logarithm of the new constant is evaluated in Appendix A.

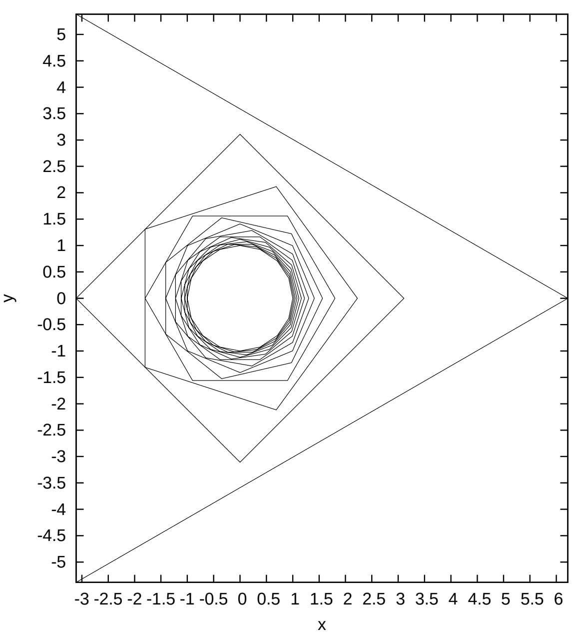


FIG. 3. Circumscribing a 16-gon by a 15-gon by a 14-gon ... down to a 3-gon, all with the same center.

### C. Prime side numbers

If $m$ is the next prime after $n$, the indices of the vertex of $n$ and edge of $m$ that describes the contact is irregular, see Fig. 4 and Table II. It is given by the pair $(j_o, j_i)$ which maximizes the value of (12). The mismatch of (13), $2j_i m - (2j_o + 1)n$ cannot be nulled for odd primes $m$ and $n$ because $2j_i m$ is even and $(2j_o+1)n$ is odd. But the value can apparently be forced to $\pm 1$ (where the sign is not important because this is an argument to the even function of the cosine), as demonstrated in Table II. [The value of Eq. (13) is either $+1$ or $-1$ depending on whether the odd number $(2j_o - 2j_i + 1)n$ is to be incremented or decremented to reach a multiple of 4, since the prime gaps $d$ are even and the values of $2j_i d$ are multiples of 4.]

Assuming this $\pm 1$ heuristics if always correct, the circumcircle radius in Figure 4 grows to

$$
\begin{aligned}
\frac{r_\infty^{(o)}}{r_3^{(o)}} &= \frac{\cos\frac{\pi}{nm}}{\cos\frac{\pi}{m}} = \frac{\cos\frac{\pi}{3\cdot 5}}{\cos\frac{\pi}{5}} \times \frac{\cos\frac{\pi}{5\cdot 7}}{\cos\frac{\pi}{7}} \times \frac{\cos\frac{\pi}{7\cdot 11}}{\cos\frac{\pi}{11}} \times \cdots \\
&= \frac{1}{2}\frac{1}{\cos\frac{\pi}{3}} \frac{\cos\frac{\pi}{3\cdot 5}}{\cos\frac{\pi}{5}} \times \frac{\cos\frac{\pi}{5\cdot 7}}{\cos\frac{\pi}{7}} \times \frac{\cos\frac{\pi}{7\cdot 11}}{\cos\frac{\pi}{11}} \times \cdots \\
&= \frac{1}{2K'_p} \prod_{p_j \ge 3} \cos\frac{\pi}{p_j p_{j+1}} \approx 1.5550895739\ldots \qquad (22)
\end{aligned}
$$

where

$$
K'_p = \prod_{p=3,5,7,11\ldots} \cos\frac{\pi}{p} \approx 0.3128329 \qquad (23)
$$

is Kitson's product over odd primes $p$ [8, 9][3, A131671]. The infinite product of cosines in (22) is evaluated in (B8) and smaller than unity, so the equation says that our construction squeezes the circumradius of the casting prime-sided regular polygons by more than a factor two compared to Kitson's variant of construction.

Fig. 4 illustrates why: A non-zero mismatch angle reflects that no vertex of the inner polygon touches a midpoint of a side of the outer polygon; in consequence the circumradius of the inner polygon is larger than the inradius of the outer polygon for each individual pair of polygons.

If polygons with sides of odd prime numbers are stacked in *reverse* order, inscribing a 5-gon in a 3-gon, a 7-gon in the 5-gon, a 11-gon in the 7-gon etc., there is no substantial modification to the calculation, because interchanging the values of $n$ and $m$ in Table II appears to lead again to a list of $\pm 1$ in the mismatches. Now the ratio of the circumradius of the triangle divided by the radius of the circular inner hole is

$$
\begin{aligned}
\frac{r_3^{(o)}}{r_\infty^{(o)}} &= \cdots \times \frac{r_7^{(o)}}{r_{11}^{(o)}} \times \frac{r_5^{(o)}}{r_7^{(o)}} \times \frac{r_3^{(o)}}{r_5^{(o)}} \\
&= \frac{\prod_{p_j \ge 3} \cos\frac{\pi}{p_j p_{j+1}}}{\prod_{p_j \ge 3} \cos\frac{\pi}{p_j}} = \frac{1}{K'_p} \prod_{p_j \ge 3} \cos\frac{\pi}{p_j p_{j+1}}, \qquad (24)
\end{aligned}
$$

so there is a straight factor of 2 relative to the value in (22).

| $n$ | $m$ | $j_i$ | $j_o$ | $2j_i m - (2j_o+1)n$ |
|---|---|---|---|---|
| 3 | 5 | 1 | 1 | 1 |
| 5 | 7 | 1 | 1 | -1 |
| 7 | 11 | 1 | 1 | 1 |
| 11 | 13 | 3 | 3 | 1 |
| 13 | 17 | 5 | 6 | 1 |
| 17 | 19 | 4 | 4 | -1 |
| 19 | 23 | 7 | 8 | -1 |
| 23 | 29 | 2 | 2 | 1 |
| 29 | 31 | 7 | 7 | -1 |
| 31 | 37 | 13 | 15 | 1 |
| 37 | 41 | 14 | 15 | 1 |
| 41 | 43 | 10 | 10 | -1 |
| 43 | 47 | 16 | 17 | -1 |
| 47 | 53 | 4 | 4 | 1 |
| 53 | 59 | 22 | 24 | -1 |
| 59 | 61 | 15 | 15 | 1 |
| 61 | 67 | 5 | 5 | -1 |
| 67 | 71 | 25 | 26 | -1 |
| 71 | 73 | 18 | 18 | 1 |

TABLE II. Vertex and edge indices $j_i$ and $j_o$ that maximize (12) for adjacent primes $n$ and $m$, describing the concentric polygons in Fig. 4.

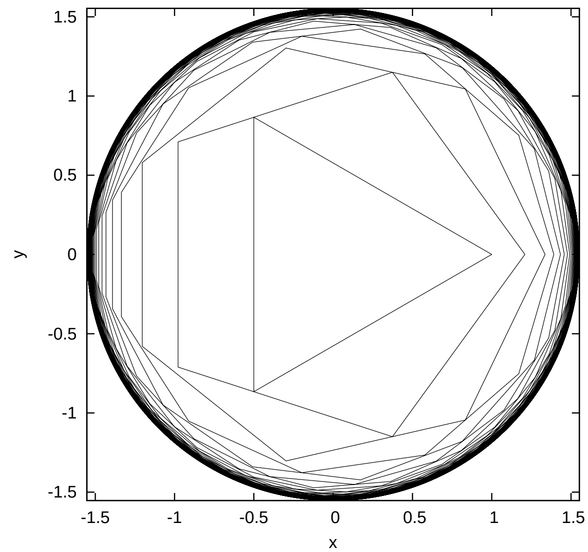


FIG. 4. Circumscribing a 3-gon by a 5-gon by a 7-gon by a 11-gon etc up to a 541-gon, all with the same center, using all odd primes as edge numbers.

## III. CONCENTRIC, ROTATIONS ALLOWED

If the outer polygon is rotated by an angle $\alpha_m$ relative to the standard position (6), the vertices move to

$$x + iy = r_m^{(o)} e^{2\pi j i/m+\alpha_m}, \quad 0 \le j < m. \tag{25}$$

In consequence, all three factors on the left hand side of (7) are multiplied by $e^{i\alpha_m}$, Eq. (8) obtains an additional factor $e^{-i\alpha_m}$ on the right hand side, and the phase shift finally enters Eq. (12):

$$\frac{r_m^{(o)}}{r_n^{(o)}} = \frac{\cos[j_i\phi_n - (j_o+1/2)\phi_m - \alpha_m]}{\cos(\phi_m/2)}. \tag{26}$$

The tightest solution for fixed $\alpha_m$ is represented by the pair $(j_i, j_0)$ which maximizes the value of $r_m^{(o)}/r_n^{(o)}$ and maximizes the value of the cosine in the numerator (because $m$ and the denominator are fixed). Shifts of $\alpha_m$ induce reduction of some peaks and rises of others in the bi-periodic domain spanned by the $j_i$ and $j_o$. The best solution is obtained where the value of the cosine becomes degenerate with highest multiplicity on the grid of the $(j_i,\ j_o)$. In geometrical terms, rotated solutions seek to maximize the number of contact points between the two polygons, illustrated in Fig. 5 and 6.

The interesting range is $0 \le \alpha_m \le \min(\phi_m, \phi_n)$, because rotation of the inner polygon by integer multiples of $\phi_n$ or rotation of the outer polygon by integer multiples of $\phi_m$ leaves the graph invariant. [Or, formally speaking, changes of $\alpha_m$ modulo $\phi_n$ or modulo $\phi_m$ can be absorbed into resetting the integers $j_i$ or $j_o$ in the numerator.] The ratio $r_m^{(o)}/r_n^{(o)}$ exercises $m$ and also $n$ periods if $\alpha$ is turned through a full angle of $2\pi$, and contains therefore $\mathrm{lcm}(m,n)$ periods. (lcm is the least common multiple of both.)

Because $j_i\phi_n$ and $(j_o+1/2)\phi_m$ have an integer representation if measured in units of $\pi/(mn)$, because the cosine is a smooth function of its argument, because the periodicity with respect to $\alpha_m$ means its extremal values can only occur at multiples of half the period, and because $nm = \mathrm{lcm}(n,m)\gcd(n,m)$, we may encode all relevant angles as $\alpha_m \equiv s_{n,m}\pi/(mn)$ with integer-valued $s_{n,m}$. The phase angle in the numerator of (26) becomes

$$\frac{\pi}{nm}[2j_i m - (2j_0+1)n - s_{n,m}]. \tag{27}$$

Investigation all possible pairs of polygons up to the 88-gon leads to the following heuristics:

- If $n$ is odd, then $s_{n,m} = 0$. [Interpretation: the mismatch (13) is odd; no vertex points exactly to the center of an edge. This establishes the following stability/frustration argument: By the up-down symmetry of the graph, infinitesimal rotation of the inner polygon requires pushing at least that edge of the outer polygon outwards, which necessarily growth in size instead of shrinking as requested.] This implies that neither the polygon pair in Figure 1 nor the cascaded stack with the primal edges numbers in Figure 4 can be compressed by adding rotations.

- Periodicity: $s_{n,m} = s_{n,m+n}$. This seems to be a consequence of the modular property mentioned

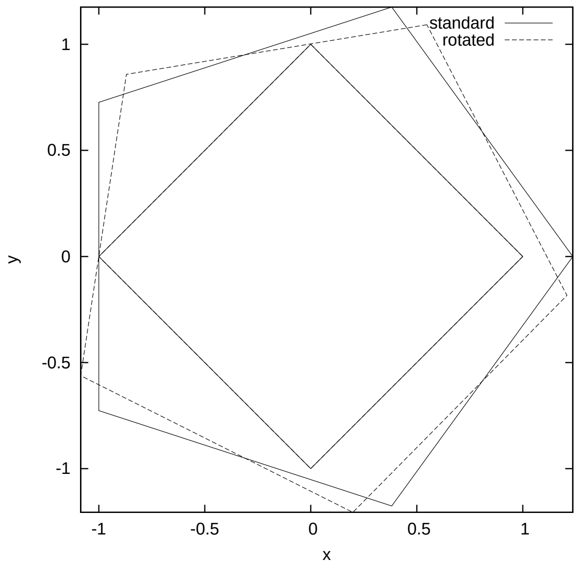


FIG. 5. Tight 5-gon around the 4-gon in the standard placement, where the angle $\alpha_5$ in (25) is kept at zero, and another—tighter—solution where $\alpha_5$ is set to $-\pi/20 = -9^\circ$ to yield a smaller 5-gon. The standard solution generates $r_5^{(o)}/r_4^{(o)} \approx 1.236$—see Table I— whereas the solution allowing rotation yields $r_5^{(o)}/r_4^{(o)} \approx 1.222$—see Table III.

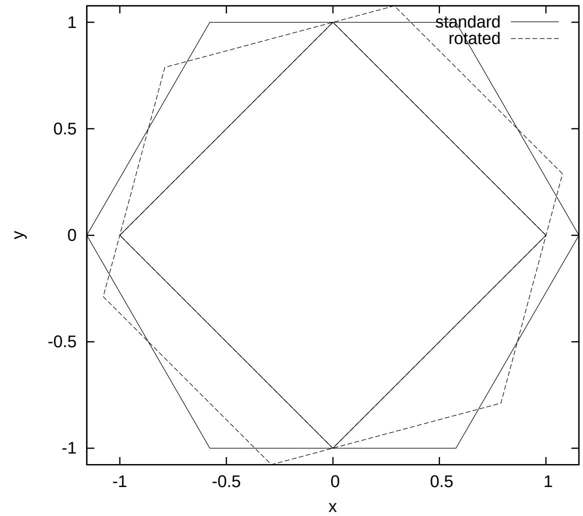


FIG. 6. Tight 6-gon around the 4-gon in the standard placement [where the angle $\alpha_6$ in (25) is kept at zero], and another tighter solution where $\alpha_6$ is set to $\pi/12 = 15^\circ$ to yield a smaller 6-gon. The standard placement achieves $r_6^{(o)}/r_4^{(o)} \approx 1.154$ according to Table I, and the version allowing rotation achieves $r_6^{(o)}/r_4^{(o)} \approx 1.115$ reported in Table III.

above; a change of $m$ by a multiple of $n$ is absorbed by modifying $j_i$ or $j_o$ by integer units. The $\gcd(n, m)$ (the period length of the cosine) is also preserved. Both aspects combined seem to freeze the number of the contacts between the two polygons.

- $s_{n,n} = 0$. If the edge numbers are equal, the circumscribed polygon is a copy of the inscribed polygon.
- If $n$ is even,
  - $s_{n,n/2} = n/2$. This says that an outer polygon with half as many vertices as the inner polygon may be constructed by outwards extension of one over the other edge of the inner polygon (which requires a rotation by half of the angle $\phi_m$ relative to the standard position). This achieves $r_m^{(i)} = r_n^{(i)}$.
  - A half period exists with palindromic symmetry: $s(n, n/2 + k) = s(n, n/2 - k)$. Reason: The mirror symmetry of the standard placement leads to equivalent solutions if the outer polygon is rotated either clockwise or counterclockwise. Solutions are *even* functions of $\alpha_m$, so sign flips of $s_{n,m}$ are irrelevant. The half period then results from a general property of (Fourier series of) periodic even functions.

Consuming these rules, we need to tabulate the $s_{n,m}$ only in the triangle of even $n$ with $0 \le m \le n/2$ for a full coverage. Then

- If $m \le n/2$ is odd and $n$ is even, then $s(n, m) = \gcd(n/2, m)$.
- If $m < n/2$ is even, and
  - $n$ is two times an odd number, $s_{n,m} = 0$.
  - $n$ is two times an even number, $s_{n,m} = 2\gcd(n/2, m/2)$. This selection is apparently aligning edge 0 of the circumscribing polygon parallel to edge 0 of the inscribed polygon with the aim to increase the number of contacts to a multiple of four, similar to what is observed in Fig. 6.

As an application, the cumulative wrench angle of the vertex direction of the outermost regular polygon in Figure 7 relative to its position in Figure 2 is calculated as

$$\sum_{n\ge 3, m=n+1} \pi \frac{s_{n,m}}{nm} = \pi \sum_{n=2,3,4,\ldots} \frac{s_{2n,2n+1}}{2n(2n+1)}$$
$$= \pi \sum_{n=2,3,4,\ldots} \frac{s_{2n,1}}{2n(2n+1)} = \pi \sum_{n=2,3,4,\ldots} \frac{1}{2n(2n+1)}$$
$$= \pi[\frac{5}{6} - \log(2)] \approx 25.23^\circ \quad (28)$$

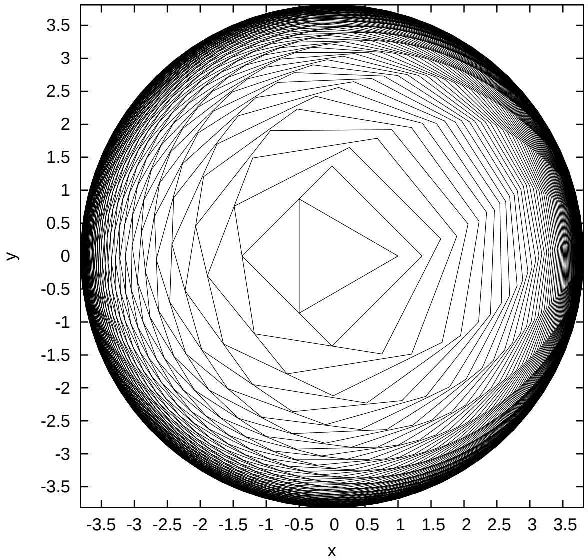


FIG. 7. Concentric encircling the 3-gon by a 4-gon by a 5-gon and so on as in Figure 2, but minimizing the areas from the 4-gon upwards by rotating these polygons by variation of $\alpha_m$.

Another heuristic observation with this rule is that the absolute value of the mismatch (13) for $m = n+1$ is kept at 1 if $n$ is odd and at 2 if $n$ is even. The growth of the radius in Figure 7 is limited to the infinite product of terms of the form (26),

$$\frac{r_\infty^{(o)}}{r_3^{(o)}} = \frac{\cos\frac{2\pi}{3\cdot 4}}{\cos\frac{\pi}{4}} \times \frac{\cos\frac{\pi}{4\cdot 5}}{\cos\frac{\pi}{5}} \times \frac{\cos\frac{2\pi}{5\cdot 6}}{\cos\frac{\pi}{6}} \times \frac{\cos\frac{\pi}{6\cdot 7}}{\cos\frac{\pi}{7}} \times \cdots$$
$$= \frac{1}{2K'} \prod_{k=3,5,7,\ldots} \cos\frac{2\pi}{k(k+1)} \prod_{k=4,6,8,\ldots} \cos\frac{\pi}{k(k+1)}$$
$$= \frac{1}{2K'\cos\frac{\pi}{2\cdot 3}} C_e \prod_{k=3,5,7,\ldots} \cos\frac{2\pi}{k(k+1)}$$
$$= \frac{1}{\sqrt{3}K'} C_e \prod_{k=3,5,7} \cos\frac{2\pi}{k(k+1)} \approx 3.5809046865583, \qquad (29)$$

where $K'$, $C_e$ and the infinite product are taken from (17), (A13) and (A25). This circumradius including rotations is considerably smaller than the circumradius (16) in the standard positions.

## IV. TRANSLATED CENTERS

A glance at Figure 2 or 5 for example reveals that further compression of the outer polygon would be possible if either one is shifted sideways, giving up the requirement that the two polygons be concentric.

In the complex plane this adds a displacement $z_m^{(o)}$ of the outer polygon as a new parameter to Eq. (7):

$$z_m^{(o)} + r_m^{(o)}\{e^{2\pi i j_o/m} + t[e^{2\pi i(j_o+1)/m} - e^{2\pi i j_o/m})]\}$$
$$= r_n^{(o)} e^{2\pi i j_i/n}. \qquad (30)$$

Assuming that $z_m^{(o)}$ is real-valued (that center shifts are sideways), this can be written as

$$\frac{z_m^{(o)}}{r_m^{(o)}}\cos(j_o\phi_m)+1+t[\cos(\phi_m)-1] = \frac{r_n^{(o)}}{r_m^{(o)}}\cos(j_i\phi_n - j_o\phi_m). \qquad (31)$$

$$-\frac{z_m^{(o)}}{r_m^{(o)}}\sin(j_o\phi_m) + t\sin(\phi_m) = \frac{r_n^{(o)}}{r_m^{(o)}}\sin(j_i\phi_n - j_o\phi_m). \qquad (32)$$

We do not discuss this parameter space systematically or solutions obtained by combined translations and rotations, but merely illustrate this aspect by the simplest examples:

- The triangle $n = 3$ could touch the quadrangle $m = 4$ in Figure 2 at its right vertex on the horizontal axis, at $j_o = j_i = t = 0$, which gives

$$\frac{z_m^{(o)}}{r_m^{(o)}} + 1 = \frac{r_n^{(o)}}{r_m^{(o)}} \qquad (33)$$

  and $0 = 0$ for the imaginary part. (This equation and overlap of the two vertices is possible whenever $m > n$.) The other two vertices of the triangle would stay glued to the quadrangle's sides, one point at $j_i = 1 = j_o$, which is

$$1 - t = \frac{r_n^{(o)}}{r_m^{(o)}}\cos(2\pi/12). \qquad (34)$$

$$-\frac{z_m^{(o)}}{r_m^{(o)}} + t = \frac{r_n^{(o)}}{r_m^{(o)}}\sin(2\pi/12). \qquad (35)$$

  The solution to these three linear equations for the three unknown $t$, $z_m^{(o)}/r_m^{(o)}$ and $r_n^{(o)}/r_m^{(o)}$ is

$$\frac{z_m^{(o)}}{r_m^{(o)}} = 1 - \frac{2}{\sqrt{3}} \approx -0.1547005, \qquad (36)$$
$$\frac{r_n^{(o)}}{r_m^{(o)}} = 2(1 - \frac{1}{\sqrt{3}}) \approx 0.84529946. \qquad (37)$$

  Therefore $r_m^{(o)}/r_n^{(o)} \approx 1.1830127$ which is indeed smaller than $r_4^{(o)}/r_3^{(o)}$ in Tables I and III.

- For $n = 4$, $m = 3$, the 3-gon circumscribing the 4-gon in Figure 3, the shift leads for the contact on the negative real line where $t = 1/2$, $j_o = (m-1)/2$, $j_i = n/2$ to

$$-\frac{1}{2}\frac{z_m^{(o)}}{r_m^{(o)}} + \frac{1}{4} = \frac{1}{2}\frac{r_n^{(o)}}{r_m^{(o)}}. \qquad (38)$$

| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 3 | 1.00000000 | 1.36602540 | 1.20905693 | 1.00000000 | 1.09751942 | 1.07313218 | 1.00000000 | 1.04570220 |
| 4 | 1.93716632 | 1.00000000 | 1.22204076 | 1.11535507 | 1.10348396 | 1.00000000 | 1.06044555 | 1.03851698 |
| 5 | 1.95629520 | 1.39680225 | 1.00000000 | 1.14837497 | 1.10544807 | 1.07905555 | 1.06158549 | 1.00000000 |
| 6 | 1.73205081 | 1.36602540 | 1.22929667 | 1.00000000 | 1.10681271 | 1.07313218 | 1.04801052 | 1.04570220 |
| 7 | 1.97766165 | 1.40532128 | 1.23109193 | 1.15147176 | 1.00000000 | 1.08068940 | 1.06285492 | 1.05040347 |
| 8 | 1.98422940 | 1.30656296 | 1.23255619 | 1.14559538 | 1.10830702 | 1.00000000 | 1.06324431 | 1.04847492 |
| 9 | 1.87938524 | 1.40883205 | 1.23305698 | 1.13715804 | 1.10853655 | 1.08136200 | 1.00000000 | 1.05082170 |
| 10 | 1.98904379 | 1.39680225 | 1.17557050 | 1.14837497 | 1.10879865 | 1.07905555 | 1.06352950 | 1.00000000 |

TABLE III. Size ratios $r_m^{(o)}/r_n^{(o)}$ as a function of the polygon edge counts for rows $n \geq 3$ and columns $m \geq 3$ for the tight concentric placements, allowing for rotations. By construction, the elements are not larger than the equivalent entries in table I.

[The generic equation if $n$ is even, $m$ is odd and $n > m$ is

$$-\frac{z_m^{(o)}}{r_m^{(o)}}\sin(\frac{\phi_m}{2}) + \frac{1}{2}\sin(\phi_m) = \frac{r_n^{(o)}}{r_m^{(o)}}\sin(\frac{\phi_m}{2}). \tag{39}$$

] Two more equations are established by $j_i = 1$, $j_o = 0$ if the upper and lower vertex of the quadrangle meets the other two edges of the triangle, namely

$$\frac{z_m^{(o)}}{r_m^{(o)}} + 1 - \frac{3}{2}t = 0; \tag{40}$$

$$\frac{\sqrt{3}}{2}t = \frac{r_n^{(o)}}{r_m^{(o)}} \tag{41}$$

such that

$$\frac{r_n^{(o)}}{r_m^{(o)}} = \frac{3}{4}(\sqrt{3}-1) \approx 1/1.821367 \tag{42}$$

with center displacement

$$\frac{z_m^{(o)}}{r_m^{(o)}} = \frac{1}{4}(5 - 3^{3/2}). \tag{43}$$

- For the 5-gon circumscribing the 3-gon in Figure 4 the shift leads for the contact on the positive real axis again to (33) plus two equations established by $j_i = j_o = 1$, such that

$$\frac{r_n^{(o)}}{r_m^{(o)}} = -\frac{2}{3} - \frac{4}{3}\cos\frac{2\pi}{5} + \frac{2}{3}\cos\frac{2\pi}{15} + 2\cos\frac{4\pi}{15} \approx 1/1.1512750 \tag{44}$$

with center displacement

$$\frac{z_m^{(o)}}{r_m^{(o)}} = -\frac{5}{3} - \frac{4}{3}\cos\frac{2\pi}{5} + \frac{2}{3}\cos\frac{2\pi}{15} + 2\cos\frac{4\pi}{15}. \tag{45}$$

## V. SUMMARY

We have defined and computed the smallest ratio of the circumradii of a pair of non-overlapping concentric regular polygons, and have pointed at infinite products of cosines that arise if some infinite sets of regular polygons are nested defined by simple strides in the sets of side numbers.

## Appendix A: Quenching Factor of the Kepler-Bouwkamp Constant

### 1. Even lower term in the product

The constant (21) is approached by calculating its logarithm (and including one more term to put the result into a more general perspective) [10],

$$\log \prod_{n=2,4,6,8,\ldots} \cos\frac{\pi}{n(n+1)} = \sum_{n=2,4,6,8,\ldots} \log\cos\frac{\pi}{n(n+1)} \approx -0.160923373349205036366901529 \tag{A1}$$

via the associated Taylor series [3, A046991][11, 1.518]

$$\log\cos\epsilon = -\frac{\epsilon^2}{2} - \frac{\epsilon^4}{12} - \frac{\epsilon^6}{45} - \frac{17\epsilon^8}{2520} - \frac{31\epsilon^{10}}{14175} - \frac{691\epsilon^{12}}{935550} - \cdots \tag{A2}$$

as follows:

$$-\log \prod_{n=2,4,6,8,\ldots} \cos\frac{\pi}{n(n+1)} = \sum_{k=1}^{\infty}[\frac{\pi^2}{2(2k)^2(2k+1)^2} + \frac{\pi^4}{12(2k)^4(2k+1)^4} + \frac{\pi^6}{45(2k)^6(2k+1)^6} + \frac{17\pi^8}{2520(2k)^8(2k+1)^8} + \cdots] \tag{A3}$$

Partial fraction decompositions of the individual terms have the following format [11, 2.102] [12–15]

$$\frac{1}{n^2(n+1)^2} = \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)}, \tag{A4}$$

$$\frac{1}{n^4(n+1)^4} = \frac{1}{n^4} + \frac{1}{(n+1)^4} - \frac{4}{n^3} + \frac{4}{(n+1)^3} + \frac{10}{n^2} + \frac{10}{(n+1)^2} - \frac{20}{n(n+1)}, \quad \text{(A5)}$$

$$\frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left[\frac{(-)^t}{n^t} + \frac{1}{(n+1)^t}\right]. \quad \text{(A6)}$$

Sums of reciprocal powers of the even or odd integers are in terms of Riemann's $\zeta$-function [11, 0.233][16, (335)]

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^t} = \frac{1}{2^t}\zeta(t), \quad \text{(A7)}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^t} = [1-\frac{1}{2^t}]\zeta(t) - 1. \quad \text{(A8)}$$

Combining the previous three equations generates (with a little extra care at $t=1$ [11, 0.234])

$$\begin{aligned}
T_e(2s) &\equiv \sum_{k=1}^{\infty} \frac{1}{(2k)^{2s}(2k+1)^{2s}} \\
&= \sum_{k=1}^{\infty}\sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left[\frac{(-)^t}{(2k)^t} + \frac{1}{(2k+1)^t}\right] \\
&= \sum_{k=1}^{\infty} \Bigg\{ -\frac{\binom{4s-2}{2s-1}}{(2k)(2k+1)} \\
&\quad + \sum_{t=2}^{2s} \binom{4s-t-1}{2s-1} \left[\frac{(-)^t}{(2k)^t} + \frac{1}{(2k+1)^t}\right] \Bigg\} \\
&= -\binom{4s-2}{2s-1}[1-\ln 2] \\
&\quad + \sum_{k=1}^{\infty}\sum_{t=2}^{2s} \binom{4s-t-1}{2s-1} \left[\frac{(-)^t}{(2k)^t} + \frac{1}{(2k+1)^t}\right] \\
&= \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left\{-1 + [1 - \frac{1-(-)^t}{2^t}]\zeta(t)\right\}. \quad \text{(A9)}
\end{aligned}$$

For odd $t$, $[1-\frac{1-(-)^t}{2^t}]\zeta(t)$ equals Dirichlet's $\eta$-function, in particular $\eta(1) = \log 2$ at the pole of $\zeta(1)$ [4, Tab. 23.3].

The three base examples of this format are:

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2(2k+1)^2} = -3 + \frac{\pi^2}{6} + 2\log 2 \approx 0.03122842796811705530687941; \quad \text{(A10)}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4(2k+1)^4} = -35 + \frac{\pi^4}{90} + 3\zeta(3) + \frac{5\pi^2}{3} + 20\log 2 \approx 0.00077822287109160078401223; \quad \text{(A11)}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^6(2k+1)^6} = -462 + 21\pi^2 + 42\zeta(3) + 252\log 2 + \frac{7\pi^4}{30} + \frac{\pi^6}{945} + \frac{45}{8}\zeta(5) \approx 0.0000214492855159526203348. \quad \text{(A12)}$$

Interchange of the summation over the Taylor orders and over the $k$ and insertion of (A9) into (A3) leads to the value (A1). Exponentiation gives

$$C_e \equiv \prod_{n=2,4,6,\ldots}^{\infty} \cos\frac{\pi}{n(n+1)} \approx 0.85135730526671405636170. \quad \text{(A13)}$$

### 2. Odd lower term in the product

In (A1), the smaller factor in the product $n(n+1)$ was always even. With exactly the same technique we obtain a "complete" version of (A9) where the smaller term steps through all positive integers:

$$T(2s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left\{[1+(-)^t]\zeta(t) - 1\right\}. \quad \text{(A14)}$$

Here $[1+(-1)^t]\zeta(t)$ is to be interpreted as 0 if $t=1$, ignoring the pole of $\zeta$. The three basic examples are

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2}{3} - 3 \approx 0.2898681336964528729448303333; \quad \text{(A15)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4(n+1)^4} = -35 + \frac{10\pi^2}{3} + \frac{\pi^4}{45} \approx 0.0633278043868051124803107260; \quad \text{(A16)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6(n+1)^6} = -462 + 42\pi^2 + \frac{7\pi^4}{15} + \frac{2\pi^6}{945} \approx 0.0156467855897643141498131091. \quad \text{(A17)}$$

The complete version of (A1) does not exist because the term at $n=1$ contributes $\log\cos(\pi/2) = -\infty$. We drop this term at $n=1$ and use $T(2s) - 2^{-2s}$ with (A2) to compute

$$\log \prod_{n=2}^{\infty} \cos\frac{\pi}{n(n+1)} \approx -0.2039684236116246918364049, \quad \text{(A18)}$$

and its exponential value

$$C \equiv \prod_{n=2}^{\infty} \cos\frac{\pi}{n(n+1)} \approx 0.8154881209503708483443 87. \quad \text{(A19)}$$

If the smaller factor in $n(n+1)$ is odd, the difference is involved:

$$T_o(2s) \equiv \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^{2s}(n+1)^{2s}} = T(2s) - T_e(2s). \tag{A20}$$

Division of (A19) through (A13) yields the complement

$$C_o \equiv \prod_{n=3,5,7\ldots}^{\infty} \cos\frac{\pi}{n(n+1)} = \frac{C}{C_e}$$
$$\approx 0.95786823687957188013580826171688 \tag{A21}$$

for use in (16).

### 3. Numerator $2\pi$ with odd lower term

The factor $2\pi$ in the numerator of (29),

$$\prod_{k=3,5,7\ldots} \cos\frac{2\pi}{k(k+1)} = \prod_{n=1}^{\infty} \cos\frac{\pi}{(2n+1)(n+1)} \tag{A22}$$

causes slow convergence of the methods shown above. An acceleration method with deferred summation is available [17]: The partial product up to some $n \le M$ is calculated explicitly, and the logarithm of the remaining infinite product is expanded in a Taylor series in $1/n$:

$$\sum_{n>M} \log\cos\frac{\pi}{(2n+1)(n+1)}$$
$$= \sum_{n>M} [-\frac{\pi^2}{8}\frac{1}{n^4} + \frac{3\pi^2}{8}\frac{1}{n^5} - \frac{23\pi^2}{32}\frac{1}{n^6} + \frac{9\pi^2}{8}\frac{1}{n^7} + \cdots] \tag{A23}$$

Each term on the right hand side is then replaced by an incomplete $\zeta$-function,

$$\sum_{n>M} \frac{1}{n^s} = \zeta(s) - \sum_{n=1}^{M} \frac{1}{n^s}. \tag{A24}$$

With $M = 10$ and (A23) expanded up to $O(n^{-30})$ we obtain for example

$$\prod_{k=3,5,7\ldots} \cos\frac{2\pi}{k(k+1)} \approx 0.83737586804154810800047 75. \tag{A25}$$

### Appendix B: Quenching Factor of Kitson's Constant

The logarithm of the product in (22) is the a sum over all odd primes $p_j \ge 3$:

$$\log \prod_{p_j\ge 3} \cos\frac{\pi}{p_j p_{j+1}} = \sum_{p_j \ge 3} \log\cos\frac{\pi}{p_j p_{j+1}} \tag{B1}$$

Again with (A2) we evaluate

$$T(2s) \equiv \sum_{p_j} \frac{1}{p_j^{2s} p_{j+1}^{2s}} \tag{B2}$$

for integer $s$.

$$T(2) \approx 0.005519522774559; \tag{B3}$$
$$T(4) \approx 0.0000204508599535; \tag{B4}$$
$$T(6) \approx 0.000000088340410739027; \tag{B5}$$
$$T(8) \approx 0.00000000039062931254965 1477, \tag{B6}$$

so

$$\sum_{p_j\ge 3} \log\cos\frac{\pi}{p_j p_{j+1}} \approx -0.02740567 \tag{B7}$$

and after exponentiation

$$\prod_{p_j\ge 3} \cos\frac{\pi}{p_j p_{j+1}} \approx 0.97296645413462553609381 92 \tag{B8}$$


[1] S. R. Finch, *Mathematical Constants*, Encyclopedia of Mathematics and its Applications No. 94 (Cambridge University Press, Cambridge, 2003).
[2] E. W. Weisstein, ed., *CRC Concise Encyclopedia of Mathematics*, 2nd ed. (Chapman & Hall/CRC, 2002).
[3] N. J. A. Sloane, Notices Am. Math. Soc. **50**, 912 (2003), http://oeis.org/, arXiv:math.CO/0312448.
[4] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, 9th ed. (Dover Publications, New York, 1972).
[5] J. H. Conway, C. Radin, and L. Sadun, arXiv:math-ph/9812019 (1998).
[6] K. Girstmair, Acta Arithm. **81**, 387 (1997).
[7] C. J. Grimstone, Math. Gaz. **64**, 120 (1980).
[8] A. R. Kitson, arXiv:math.HO/0608186 (2006).
[9] A. R. Kitson, Math. Gaz **92**, 293 (2008).
[10] E. Stephens, Math. Gaz. **79**, 561 (1995).
[11] I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed. (Harri Deutsch, Thun, 1981).
[12] J. J. Mahoney and B. D. Sivazlian, J. Comp. Appl. Math. **9**, 247 (1983).
[13] D. J. Velleman, Am. Math. Monthly **109**, 746 (2002).
[14] G. Xin, arXiv (2004), arXiv:math/0408189.
[15] D. Eustice and M. S. Klamkin, Am. Math. Monthly **86**, 478 (1979).

[16] J. B. W. Jolley, ed., *Summation of series*, 2nd ed. (Dover Publications, New York, 1961).
[17] P. Sebah and X. Gourdon, "Constants from number theory," (2001), http://numbers.computation.free.fr/Constants/constants.html.